New Calabi-Yau Threefolds From Free Quotients and Topological Transitions

Rhys Davies

Mathematical Institute,
University of Oxford

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Outline

Motivation

Constructing free quotients

New from old I — Conifold transitions

New from old II — Hyperconifold transitions

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Heterotic compactifications

- Standard model gauge group is

$$SU(3)\times SU(2)\times U(1)_Y \;\subset\; SU(5) \;\subset\; SU(5)\times SU(5) \;\subset\; E_8 \;,$$

  so centraliser is $U(1)_Y \times SU(5)$.

- But $U(1)_Y$ flux implies a massive gauge boson.
- Solution: Discrete Wilson lines

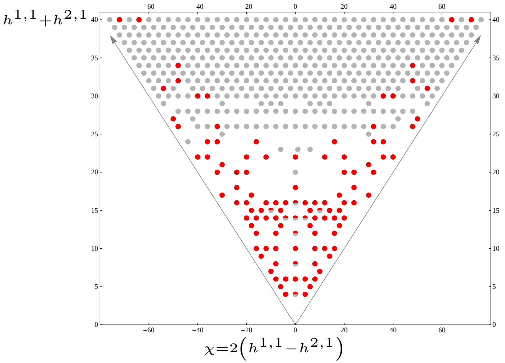
  $\Longrightarrow$ non-trivial fundamental group.

Moduli stabilisation

- Fewer moduli may make stabilisation easier/more tractable.
- E.g. Anderson et. al. "Stabilizing All Geometric Moduli in Heterotic Calabi-Yau Vacua", **arXiv:1102.0011**:
  - Uses $h^{1,1}(X) - 1$ line bundles.
  - Only possible for $h^{1,1}(X) \lesssim 10$.
  - Conditions on $h^{2,1}$ less clear.

Hodge numbers and fundamental group

- $\pi_1(X) \neq \mathbf{1}$ implies torsion in (co)homology. Mirror symmetry preserves torsion, although not π_1 itself.



- Torsion-free (co)homology.
- Torsion in (co)homology.
- At least one manifold with each property.
- Other.

- Red dots with $\chi \leq 0$ have $\pi_1 \neq \mathbf{1}$, others are mirrors.

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Calabi-Yau covering spaces

- Define 'Calabi-Yau' (CY) as Kähler with $c_1 = 0$ (over $\mathbb{Z}$). Notation:
  - ω is the Kähler form — a closed positive $(1,1)$-form.

$$\text{Positivity: } \int_S \omega^n > 0 \;\; \forall \; \text{complex sub-manifolds } S$$

  - Ω is the nowhere-zero holomorphic $(3,0)$-form.
- Every manifold has a universal cover. Suppose $X = \widetilde{X}/G$ is Calabi-Yau, where G acts freely.

  Then so is $\widetilde{X}$, since if $\pi : \widetilde{X} \to X$ is the covering map,
  - $d(\pi^*\omega) = \pi^*(d\omega) = 0$, and $\pi^*\omega$ also positive.
  - $\pi^*\Omega$ is a nowhere-zero holomorphic $(3,0)$-form (check pointwise).

Calabi-Yau quotient spaces

- What about the converse? Let G act freely, holomorphically on $\widetilde{X}$. Is $X = \widetilde{X}/G$ Calabi-Yau?

- Choose any Kähler form ω on $\widetilde{X}$. Then

$$\omega^G := \sum_{g \in G} g^* \omega$$

is a G-invariant Kähler form, so descends to a Kähler form on X.

Calabi-Yau quotient spaces

- Ω is unique (up to scale) element of $H^{3,0}(\widetilde{X})$. Since G acts without fixed points, an Atiyah-Bott fixed point formula reduces to

$$0 \;=\; \sum_{q=0}^{3} (-1)^q \operatorname{Tr}\left(g^*\big|_{H^{3,q}}\right) \;=\; \operatorname{Tr}\left(g^*\big|_{H^{3,0}}\right) - \operatorname{Tr}\left(g^*\big|_{H^{3,3}}\right)$$

  for any $g \in G \setminus e$.
- But $H^{3,3}(X)$ is spanned by $\left(\omega^G\right)^3$, which is invariant, so $\operatorname{Tr}\left(g^*\big|_{H^{3,3}}\right) = 1$.
- We conclude that $g^*\Omega = \Omega$, so Ω descends to a nowhere-zero holomorphic $(3,0)$-form on X.

Calabi-Yau quotient spaces

- Conclusion: If a group G acts holomorphically without fixed points on a Calabi-Yau threefold $\widetilde{X}$, then $X = \widetilde{X}/G$ is automatically Calabi-Yau.

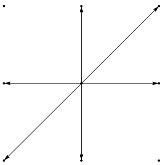
- Note: Argument holds in all odd dimensions, and also shows that in even dimensions, all Calabi-Yau manifolds are simply connected.

Example: Quotients of CICY's

- All free quotients of complete intersection CY's in products of projective spaces are classified: Braun, **arXiv:1003.3235**.
- One finds a linear group action on the ambient space, then checks:
  - The symmetric CY sub-manifolds are generically smooth.
  - They do not intersect the fixed-point set in the ambient space.
- Plenty of details and examples in Candelas, Davies **arXiv:0809.4681**.

Example: New three-generation manifolds

- $X^{8,44}$, a CICY, but also hypersurface in $\mathrm{dP}_6 \times \mathrm{dP}_6$.
- Fan for dP_6:



- Hexagonal! So $\mathrm{dP}_6 \times \mathrm{dP}_6$ has symmetry $(D_6 \times D_6) \rtimes \mathbb{Z}_2$.
- Two order-12 subgroups act freely on $X^{8,44}$: $\mathbb{Z}_{12}$, $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$.
- The quotients have $(h^{1,1}, h^{2,1}) = (1, 4)$ and thus $\chi = -6$, giving three generations by standard embedding.

  Many details in Braun, Candelas, Davies, **arXiv:0910.5464**.

  Symmetry breaking in my thesis: http://people.maths.ox.ac.uk/daviesr

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The conifold

- Simplest singularity of a complex threefold:

$$y_1 y_4 - y_2 y_3 = 0 \ .$$

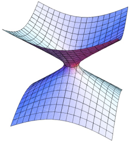
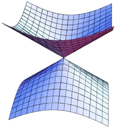
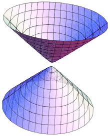
  (Generally, a co-dimension k space given by $f_1 = \ldots = f_k = 0$ is singular where $df_1 \wedge \ldots \wedge df_k = 0$ also holds.)

- Its topology is a cone over $S^3 \times S^2$.

  See e.g. Candelas and de la Ossa, **Nucl.Phys. B342 (1990) 246-268**

- This singularity can be *deformed* or *resolved*.

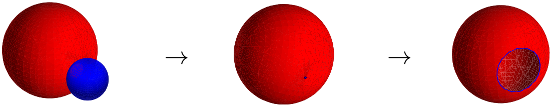
Conifold transitions locally

	Deformation	Conifold	Resolution
Geometry			
Equation	$y_1y_4 - y_2y_3 = \epsilon$	$y_1y_4 - y_2y_3 = 0$	$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- Deformation replaces singular point with an S^3, resolution with an S^2.

Conifold transitions globally

- Suppose a CY manifold X deforms to X_0, with conifold singularities.
- In resolving X_0, must be careful about Kähler condition:
  - E.g. Suppose $A \cong S^3$ vanishes and is replaced by $C \cong S^2$.
  - Dual cycle B with $B \cap A = 1$. Then after resolution, $C = \partial B$.



  - Then a Kähler form ω must satisfy

$$0 = \int_B d\omega = \int_{\partial B} \omega = \int_C \omega > 0 \text{ . Impossible!}$$

- Condition: Non-trivial homology relations between vanishing S^3's.

Conifold transitions globally

- Useful fact: for algebraic manifolds, Kähler $\Leftrightarrow$ Projective.
- Example: $X^{1,101} \rightsquigarrow X^{2,86}$
  - $\mathbb{P}^4$ with homogeneous coordinates $z_0, \ldots, z_4$. Special quintics:

$$z_0\, g_0(z) - z_1\, g_1(z) = 0 \ .$$

  - All contain $\{z_0 = z_1 = 0\} \cong \mathbb{P}^2$. Since g_0, g_1 are quartics, this family is singular at $4{\times}4 = 16$ points, all lying in this $\mathbb{P}^2$.
  - Introduce a $\mathbb{P}^1$, and consider in $\mathbb{P}^1{\times}\mathbb{P}^4$

$$\begin{pmatrix} z_1 & -z_0 \\ g_0 & -g_1 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- This construction is called "blowing up" along the $\mathbb{P}^2$. The blow-up of a projective variety is always projective.

Conifold transitions and π_1

- Suppose we have $\widetilde{X}/G =: X \rightsquigarrow Y$. What is $\pi_1(Y)$?

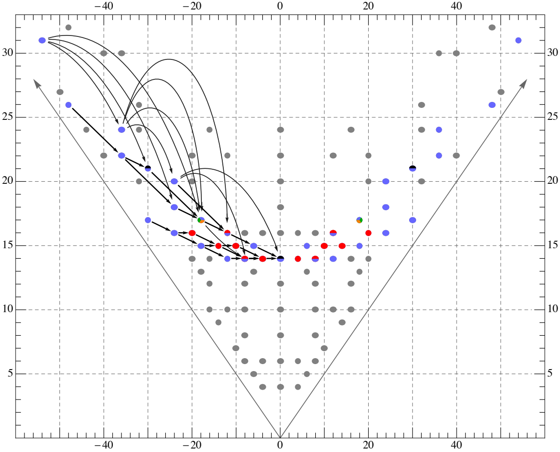
  π_1 is topological, so use 'surgery' picture to calculate:

  - Shrinking S^3's on X; delete a n'hd of each, with boundary $S^3 \times S^2$.
  - Now glue in a 'fat' S^2, with boundary $S^3 \times S^2$, in place of each S^3.
  - S^3, $S^3 \times S^2$, S^2 all simply-connected, so we get* $\pi_1(Y) = \pi_1(X)$.

- Conclusion: Conifold transitions do not change π_1.
- So find new manifolds with $\pi_1 \neq \mathbf{1}$ by transitions from old ones!

*This follows from a simple application of van Kampen's theorem.
See e.g. Hatcher, *"Algebraic Topology"*.

Example: The $\mathbb{Z}_3$ web



- Taken from Candelas and Constantin, **arXiv:1010.1878**.

  Many of these described in Candelas and Davies, **arXiv:0809.4681**.

Example: $X^{1,4} \rightsquigarrow X^{2,2}$

- Recall $X^{1,4} = X^{8,44}/G$ where $|G| = 12$. We can embed $X^{8,44}$ as

$$X^{8,44} = \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \left[\begin{array}{ccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array}\right]$$

  Take homogeneous coordinates $x_{\alpha,j}$, $\alpha = 1,2,3,4$, $j = 0,1,2$.

  $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ generated by

$$g_3 \ : \ x_{\alpha,j} \to \zeta^{(-1)^\alpha j} x_{\alpha,j} \ , \quad g_4 \ : \ x_{\alpha,j} \to x_{\alpha+1,j}$$

  with $\zeta = \exp(2\pi \mathrm{i}/3)$.

- All the action is in the invariant degree-four polynomial r.

Example: $X^{1,4} \rightsquigarrow X^{2,2}$

- In the same way as for the quintic, special choices of r 'factorise':

$$r = f_0(x_1, x_3)g_0(x_2, x_4) - f_1(x_1, x_3)g_1(x_2, x_4)$$

- This gives 36 conifolds on $X^{8,44}$, and 3 on the quotient.
- Resolve by introducing a $\mathbb{P}^1$ with coordinates t_0, t_1,

$$t_0 f_1 - t_1 f_0 = t_0 g_0 - t_1 g_1 = 0\ .$$

- The configuration matrix is now

$$X^{19,19} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \end{bmatrix}$$

Example: $X^{1,4} \rightsquigarrow X^{2,2}$

$$t_0 f_1 - t_1 f_0 = t_0 g_0 - t_1 g_1 = 0$$

- Can deduce group action on t_0, t_1 from that on other coordinates.

$$X^{19,19} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \end{bmatrix} \cong \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix}$$

- In the second form, $X^{19,19}$ was known to admit free quotients by

$$\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_4,\ \mathbb{Z}_5,\ \mathbb{Z}_6,\ \mathbb{Z}_4 \times \mathbb{Z}_2,\ \mathbb{Z}_3 \times \mathbb{Z}_3\ .$$

(See Bouchard and Donagi, **arXiv:0704.3096**)

- Pursuing conifold transitions has revealed two more:

$$\mathbb{Z}_{12},\ \mathbb{Z}_3 \rtimes \mathbb{Z}_4\ .$$

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Aw Dad, not the quintic again! I want a G.I. Joe!

- $\mathbb{Z}_5$ naturally acts on $\mathbb{P}^4$:

$$(z_0, z_1, z_2, z_3, z_4) \to (z_0, \zeta z_1, \zeta^2 z_2, \zeta^3 z_3, \zeta^4 z_4)\ , \quad \zeta = \exp(2\pi \mathrm{i}/5)$$

- An invariant quintic hypersurface is given by

$$f \;=\; \sum_{\substack{i+j+k+l+m\equiv 0 \\ \mathrm{mod}\ 5}} A_{ijklm}\, z_i\, z_j\, z_k\, z_l\, z_m \;=\; 0\ .$$

- Fixed points when only one z_i non-zero. CY misses these if $A_{iiiii} \neq 0\ \forall\ i$.
- Symmetric hypersurfaces generically smooth, so get smooth quotients:

$$X^{1,21} = X^{1,101}/\mathbb{Z}_5\ .$$

The $\mathbb{Z}_5$-hyperconifold

- Expand f in the neighbourhood of fixed point $(1,0,0,0,0)$.

  With local coordinates $y_i = z_i/z_0$, we get (after possible rescaling)

$$f \;=\; A_{00000} + y_1\, y_4 - y_2\, y_3 + \ldots$$

- Fixed point when $A_{00000} \to 0$. Then fixed point is a conifold!
- So quotient develops 'hyperconifold' — a quotient of the conifold.

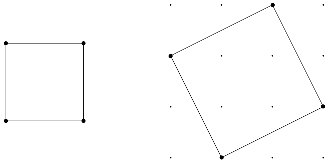


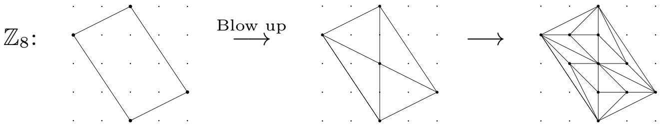
Figure: The toric diagram of the conifold and $\mathbb{Z}_5$-hyperconifold.

Hyperconifolds generally

- In **arXiv:0911.0708** I show this is a general phenomenon:
    - Let $\mathbb{Z}_N$ act *freely* on a generic member of a smooth family $\widetilde{X}$.
    - Then if a fixed point develops, it is a conifold in $\widetilde{X}$.
    - Thus the smooth family $X = \widetilde{X}/\mathbb{Z}_N$ develops a *hyperconifold*.
    - Constrast with case of generic fixed points, which give orbifolds.
- Known cases: $N = 2, 3, 4, 5, 6, 8, 10, 12$

Hyperconifold transitions

- Can we, like for some conifolds, resolve to find new manifolds?
- Yes, seemingly always! For $\mathbb{Z}_{2M}$ case, blowing up singular point gives a Calabi-Yau with only orbifold singularities.



- Since we let a fixed point develop, such a transition changes π_1.
- The Hodge numbers also change; for a $\mathbb{Z}_N$-hyperconifold transition,

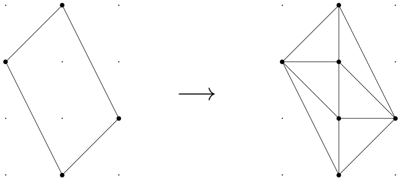
$$\delta(h^{1,1}, h^{2,1})_{\mathbb{Z}_N} = (N-1, -1)$$

Example

- Remaining cases, $\mathbb{Z}_3$ and $\mathbb{Z}_5$, shown by example to occur in Davies, **arXiv:1102.1428**.

- Family $X^{2,83} = \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ admits free $\mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ actions.

  Get $\mathbb{Z}_3$-hyperconifold transitions:



- Globally, we get $X^{2,29} \rightsquigarrow X^{4,28}$, and π_1 changes from $\mathbb{Z}_3$ to $\mathbf{1}$.

Chains of transitions

- The ambient space has nine fixed points.

  Treating them independently gives a chain of nine transitions:

  $$X^{2,29} \rightsquigarrow X^{4,28} \rightsquigarrow X^{6,27} \rightsquigarrow \ldots \rightsquigarrow X^{20,20} \; .$$

  At each step, $\delta(h^{1,1}, h^{2,1}) = (2,-1)$. Only $X^{2,29}$ has $\pi_1 \neq \mathbf{1}$.

- Can also start with $X^{2,11} = X^{2,83}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, and get

  $$X^{2,11} \rightsquigarrow X^{4,10} \rightsquigarrow X^{6,9} \rightsquigarrow X^{8,8} \; .$$

  Last three have $\pi_1 = \mathbb{Z}_3$.

- No systematic study done — possibly many new manifolds.

Conclusion

- Calabi-Yau's with few moduli and/or $\pi_1 \neq \mathbf{1}$ are particularly interesting.
- In recent years, many new such manifolds from free quotients.
- Topological transitions generate interesting new manifolds from old:
  - Conifold transitions do not change π_1.
  - Hyperconifold transitions sometimes do.